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On the degree bounds of the Graver basis

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Joint work with A.Thoma

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Basic Notations of Toric Ideals

Let $A = \{a_1, \ldots, a_m\} \subseteq \mathbb{N}^n$ be a vector configuration and $\mathbb{N}A := \{l_1a_1 + \cdots + l_ma_m \mid l_i \in \mathbb{N}\}$ the corresponding affine semigroup. We grade the polynomial ring $\mathbb{K}[x_1, \ldots, x_m]$ over an arbitrary field $\mathbb{K}$ by the semigroup $\mathbb{N}A$ setting $\deg_A(x_i) = a_i$ for $i = 1, \ldots, m$. For $u = (u_1, \ldots, u_m) \in \mathbb{N}^m$, we define the $A$-degree of the monomial $x^u := x_1^{u_1} \cdots x_m^{u_m}$ to be

$$u_1a_1 + \cdots + u_ma_m \in \mathbb{N}A.$$

We denoted by $\deg_A(x^u)$, while the usual degree $u_1 + \cdots + u_m$ of $x^u$ we denoted by $\deg(x^u)$. 
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**Definition**

The toric ideal $I_A$ associated to $A$ is the binomial ideal

$$I_A = \langle x^u - x^v : \deg_A(x^u) = \deg_A(x^v) \rangle.$$
A nonzero binomial $x^u - x^v$ in $l_A$ is called primitive if there exists no other binomial $x^w - x^z$ in $l_A$ such that $x^w | x^u$ and $x^z | x^v$.

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The set of the primitive binomials forms the Graver basis of $I_A$ and is denoted by $Gr_A$.

- The support of a monomial $x^u$ of $\mathbb{K}[x_1, \ldots, x_m]$ is $supp(x^u) := \{i \mid x_i \text{ divides } x^u\}$ and the support of a binomial $B = x^u - x^v$ is $supp(B) := supp(x^u) \cup supp(x^v)$. 
A nonzero binomial $x^u - x^v$ in $I_A$ is called **primitive** if there exists no other binomial $x^w - x^z$ in $I_A$ such that

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- An irreducible binomial $B$ belonging to $I_A$ is called a **circuit** of $I_A$ if there is no binomial $B' \in I_A$ such that $\text{supp}(B') \subsetneq \text{supp}(B)$. 

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A nonzero binomial $x^u - x^v$ in $l_A$ is called \textit{primitive} if there exists no other binomial $x^w - x^z$ in $l_A$ such that $x^w | x^u$ and $x^z | x^v$.

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- The set of the circuits is denoted by $\mathcal{C}_A$ and it is a subset of the Graver basis.
Elements of Graph Theory

- Let $G$ be a finite simple connected graph on the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and $E(G) = \{e_1, \ldots, e_q\}$ be the set of edges of $G$.

- A walk of length $s$ connecting $v_1 \in V(G)$ and $v_{s+1} \in V(G)$ is a finite sequence of the form $w = (\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_s, v_{s+1}\})$ with each $\{v_j, v_{j+1}\} \in E(G)$.

- An even (respectively odd) walk is a walk of even (respectively odd) length.

- A walk $w = (e_1 = \{v_1, v_2\}, e_2 = \{v_2, v_3\}, \ldots, e_q = \{v_s, v_{s+1}\})$ is called closed if $v_{s+1} = v_1$.

- A cycle is a closed walk $(\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_q, v_1\})$ with $v_k \neq v_j$, for every $1 \leq k < j \leq q$. 

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Elements of Graph Theory

- We denote by $w$ the subgraph of $G$ with vertices the vertices of the walk and edges the edges of the walk $w$.

- A **cut edge** (respectively **cut vertex**) is an edge (respectively vertex) of the graph whose removal increases the number of connected components of the remaining subgraph.

- A graph is called **biconnected** if it is connected and does not contain a cut vertex. A **block** is a maximal biconnected subgraph of a given graph $G$. 
Definition of $I_G$

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- We will associate each edge $e = \{v_i, v_j\} \in E(G)$ with the element $a_e = v_i + v_j$ in the free abelian group $\mathbb{Z}^n$ with basis the set of vertices of $G$. 

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**Definition**

We denote by $I_G$ the toric ideal $I_{A_G}$ in $K[e_1, \ldots, e_m]$, where $A_G = \{a_e \mid e \in E(G)\} \subset \mathbb{Z}^n$. 
Definition of $I_G$

Given an even closed walk $w = (e_1, \ldots, e_{2q-1}, e_{2q})$ of the graph $G$ we denote by $E^+(w) = q \prod_{k=1}^{2q-1} e_{2k} - 1$, $E^-(w) = q \prod_{k=1}^{2q-1} e_{2k}$, and by $B_w$ the binomial $B_w = E^+(w) - E^-(w) \in I_G$.

Theorem (Villarreal, 1995)

The toric ideal $I_G$ is generated by binomials of this form $I_G = \langle B_w : w \text{ is an even closed walk of } G \rangle$. 

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The toric ideal $I_G$ is generated by binomials of this form

$$I_G = \langle B_w, \ w \ is \ an \ even \ closed \ walk \ of \ G \rangle.$$
Example

\[ \begin{align*}
1 & \quad 2 \\
3 & \quad 4 \\
8 & \quad 7 \\
6 & \quad 5
\end{align*} \]

Therefore \( B_{w_1}, B_{w_2}, B_{w_3} \in \mathcal{I}_G \).
Example

\[ w_1 = (e_1, e_2, e_7, e_8) \] we have that \( E^+(w_1) = e_1e_7 \) and \( E^-(w_1) = e_2e_8 \) therefore \( B_{w_1} = e_1e_7 - e_2e_8 \).
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- \( w_2 = (e_3, e_4, e_5, e_6) \implies B_{w_2} = e_3e_5 - e_4e_6 \)
Example

- \( w_1 = (e_1, e_2, e_7, e_8) \) we have that \( E^+(w_1) = e_1e_7 \) and \( E^-(w_1) = e_2e_8 \) therefore \( B_{w_1} = e_1e_7 - e_2e_8 \).
- \( w_2 = (e_3, e_4, e_5, e_6) \implies B_{w_2} = e_3e_5 - e_4e_6 \)
- \( w_3 = (e_1, e_2, \ldots, e_8) \implies B_{w_3} = e_1e_3e_5e_7 - e_2e_4e_6e_8 \).

Therefore

\[ B_{w_1}, B_{w_2}, B_{w_3} \in I_G. \]
The set of the circuits of $I_G$

The following theorem determines the form of the circuits of a toric ideal of a graph $G$.

Theorem (Villareal, 1995)

Let $G$ be a graph and let $W$ be a connected subgraph of $G$. The subgraph $W$ is the graph $w$ of a walk $w$ such that $B_w$ is a circuit if and only if

1. $W$ is an even cycle or
2. $W$ consists of two odd cycles intersecting in exactly one vertex or
3. $W$ consists of two vertex-disjoint odd cycles joined by a path.
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The Graver basis of $I_G$

A walk $w$ is primitive if and only if the corresponding binomial $B_w$ is primitive.

**Example**

In the previous example, the binomial $B_{w_3} = e_1 e_3 e_5 e_7 - e_2 e_4 e_6 e_8$ is not primitive. There exists the even closed subwalk $w_1 = (e_1, e_2, e_7, e_8)$ of $w_3$, where its corresponding binomial is $B_{w_1} = e_1 e_7 - e_2 e_8$. We remark that $E^+(w_1) | E^+(w_3)$ and $E^-(w_1) | E^-(w_3)$. 
The next Theorem describes the form of the underlying graph of a primitive walk.

**Theorem (Reyes, - , Thoma, 2012)**

Let $G$ be a graph and let $W$ be a connected subgraph of $G$. The subgraph $W$ is the graph $w$ of a primitive walk $w$ if and only if

1. $W$ is an even cycle or
2. $W$ is not biconnected and
   1. every block of $W$ is a cycle or a cut edge and
   2. every cut vertex of $W$ belongs to exactly two blocks and separates the graph in two parts, the total number of edges of the cyclic blocks in each part is odd.
One of the fundamental problems in toric algebra is to give good upper bounds on the degrees of the elements of the Graver basis.
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It was conjectured that the degree of any element in the Graver basis $\mathcal{G}_A$ of a toric ideal $I_A$ is bounded above by the maximal true degree of any circuit in $C_A$.

**True Circuit conjecture (Sturmfels, 1995)**

Let us call $t_A$ the maximal true degree of any circuit in $C_A$. Then

$$\deg(B) \leq t_A,$$

for every $B \in \mathcal{G}_A$. 

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Definition (Sturmfels)

Let $C \in C_A$ be a circuit and regard its support $\text{supp}(C)$ as a subset of $A$. The index of the circuit $C$ is denoted by $\text{index}(C)$ and is defined as:

$$\text{index}(c) = [\mathbb{R}(\text{supp}(c)) \cap \mathbb{Z}A : \mathbb{Z}(\text{supp}(c))].$$

The true degree of a circuit $C$ is defined:

$$\text{true deg}(C) = \deg(C) \cdot \text{index}(C)$$
Recall:

**True Circuit conjecture**

Let us call $t_A$ the maximal true degree of any circuit in $C_A$. Then

$$\deg(B) \leq t_A,$$

for every $B \in \Gr_A$.

There are several examples of families of toric ideals where the true circuit conjecture is true. It is also true for some families of toric ideals of graphs.
Recall:

**True Circuit conjecture**

Let us call $t_A$ the maximal true degree of any circuit in $C_A$. Then

$$\text{deg}(B) \leq t_A,$$

for every $B \in \mathcal{Gr}_A$.

There are several examples of families of toric ideals where the true circuit conjecture is true. It is also true for some families of toric ideals of graphs. However the true circuit conjecture is not true in the general case [ - , Thoma, 2011].
To answer the conjecture we gave an infinite family of counterexamples by providing toric ideals of graphs such that:

\[ t_A < \deg(B) \leq (t_A)^2, \quad \forall B \in Gr_A. \]
The Problem

To answer the conjecture we gave an infinite family of counterexamples by providing toric ideals of graphs such that:

\[ t_A < \deg(B) \leq (t_A)^2, \, \forall B \in Gr_A. \]
Theorem

Let $G$ be a graph and let $C$ be a circuit in $C_{AG}$. Then

$$true \ deg(C) = deg(C).$$
Theorem

Let $G$ be a graph and let $C$ be a circuit in $C_{A_G}$. Then

$$true \ deg(C) = \ deg(C).$$

Therefore to compare the maximum degrees of primitive elements of the $l_G$ with $t_A$, its enough to compare the $\deg(B)$ with $\deg(C)$, for every $B \in Gr_A$ and for every $C \in C_A$. 
Let \( n \geq 3 \) be an odd integer. Let \( G_0^n \) be a cycle of length \( n \). For \( r \geq 0 \) we define the graph \( G_r^n \) inductively on \( r \). \( G_r^n \) is the graph taken from \( G_{r-1}^n \) by adding to each vertex of degree two of the graph \( G_{r-1}^n \) a cycle of length \( n \).
Let $n \geq 3$ be an odd integer. Let $G^n_0$ be a cycle of length $n$. For $r \geq 0$ we define the graph $G^n_r$ inductively on $r$. $G^n_r$ is the graph taken from $G^n_{r-1}$ by adding to each vertex of degree two of the graph $G^n_{r-1}$ a cycle of length $n$.

**Example**

We see in this figure the graphs $A = G^3_0$, $B = G^3_1$ and $C = G^3_2$.

We note that the graph $G^n_r$ is Eulerian since by construction it is connected and every vertex has even degree.
Proposition

Let $w^n_{r}$ be any even closed Eulerian walk of the graph $G^n_r$. The binomial $B_{w^n_r}$ is an element of the Graver basis of $I_{G^n_r}$ and

$$\deg(B_{w^n_r}) = \frac{1}{2}(n + n^2\left(\frac{(n - 1)^r - 1}{n - 2}\right)).$$
Proposition

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$$\deg(B_{w^n_r}) = \frac{1}{2}(n + n^2\left(\frac{(n - 1)^r - 1}{n - 2}\right)).$$

Proposition

Let $t_{A_{G^n_r}}$ the maximum degree of a circuit in the graph $G^n_r$. Then

$$t_{A_{G^n_r}} = n + (2r - 1)(n - 1).$$
Theorem

The degrees of the elements in the Graver basis of a toric ideal $I_A$ cannot be bounded polynomially above by the maximal true degree of a circuit.

Or equivalently:
There is not a polynomial $f(x) \in \mathbb{R}[x]$, such that

$$\deg(B) \leq f(t_A), \forall B \in Gr_A,$$

for a toric ideal $I_A$, where $t_A$ is the maximal true degree of the circuits $C \in C_A$. 

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Thank you!!!